

In the name of GOD

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$$n(t) = \operatorname{Re} \left\{ z(t) e^{j2\pi f_c t} \right\}$$

$$n(t) = x(t) \cos 2\pi f_c t - y(t) \sin 2\pi f_c t$$

$$z(t) = x(t) + j y(t)$$

$$n(t) = a(t) \cos(2\pi f_c t + \theta(t))$$

$$a(t) = \sqrt{x^2(t) + y^2(t)}$$

$$\theta(t) = \tan^{-1} \frac{y(t)}{x(t)}$$

$$\varphi_n(\tau) = \varphi_x(\tau) + 2\pi f_c \tau - \varphi_{yx}(\tau) \sin 2\pi f_c \tau$$



- { 1)  $x(t)$  &  $y(t)$  are zero-mean jointly wss stoch.  
 Processes  
 2)  $\varphi_x(\tau) = \varphi_y(\tau) \implies S_x(f) = S_y(f)$   
 3)  $\varphi_{xy}(\tau) = -\varphi_{yx}(\tau)$

$$\Phi_n(z) = \underbrace{\varphi_x(z)}_{\times} e^{jz\pi f_c z} - \varphi_{y_x}(z) \sin z\pi f_c z$$

for det. Signals we have,  $s(t) = x(t) e^{j2\pi f_c t} - y(t) \sin 2\pi f_c t$

Similarity

$$\exists \varphi_z(z) \Rightarrow \varphi_n(z) = \operatorname{Re} \left\{ \varphi_z(z) e^{jz\pi f_c z} \right\}$$

So, we should find  $\varphi_z(z)$  such that the above relation is correct.

If we define,

$$\varphi_z(z) \triangleq \frac{1}{2} \in \left\{ z(t+\tau) - z^*(\tau) \right\} \quad ①$$

So we will reach our aim. To show this fact, we will simplify the relation.

we know that ,  $z(t) = x(t) + jy(t)$  ②

①, ②  
⇒

$$\varphi_z(z) = \frac{1}{z} \in \left\{ z(t+z) z^*(z) \right\}$$

$$= \frac{1}{z} \in \left\{ [x(t+z) + j y(t+z)] \underbrace{[x(t) + j y(t)]^*} \right\}$$

$$= \frac{1}{z} \in \left\{ [x(t+z) x^*(t) + y(t+z) y^*(t)] \right\}$$

$$+ \frac{1}{z} \in \left\{ j [x(t+z) y^*(t) - y(t+z) x^*(t)] \right\}$$

$\Rightarrow$   
E.I.

is linear  
operation

$$\varphi_z(z) = \frac{1}{2} \left[ \varphi_x(z) + \underbrace{\varphi_y(z)}_{\varphi_x(z)} + \left( \varphi_{yx}(z) - \underbrace{\varphi_{xy}(z)}_{-\varphi_{yx}(z)} \right) \right]$$

$$\varphi_z(z) = \varphi_x(z) + j \varphi_{yx}(z)$$

$$\varphi_x(z) = \varphi_y(z)$$

$$\varphi_{yx}(z) = -\varphi_{xy}(z)$$

As you can see, this relation  
is similar to the relation  
we had find for deter. bandpass

Signals based on their quadrature elements.  
So we can conclude that the similar is also  
true for Correlation Function of band pass stoch. Processes

based on the corr. function of emu. low pass stoch. process.

$$\Rightarrow \varphi_n(\tau) = \operatorname{Re} \left\{ \varphi_z(z) e^{jz\pi f_c \tau} \right\}$$

So, getting the Fourier Transform, we have

$$FT \left\{ \varphi_n(\tau) \right\} = FT \left\{ \operatorname{Re} \left\{ \varphi_z(z) e^{jz\pi f_c \tau} \right\} \right\}$$

$$\Rightarrow S_n(f) = \frac{1}{2} F_T \left\{ \varphi_z(z) e^{j2\pi f_c z} + \overline{\varphi_z(z)} e^{-j2\pi f_c z} \right\}$$

$\Re\{z\} = \frac{1}{2}(z + z^*)$

$$\Rightarrow S_n(f) = \frac{1}{2} [\varphi_z(f - f_c) + \overline{\varphi_z(-f - f_c)}]$$

This relation shows the rep. of the PSD of bandpass

its  
stochastic process  $(n(t))$  based on the PSD of  $v_{\text{eqn.}}$   
low-pass stochastic process  $(z(t))$

As you know the PSD of any stock. Processes is a positive  
real function of  $f$ . So  $\underline{\Phi_z^*(-f-f_c)} = \underline{\Phi_z(-f-f_c)}$

$$\Phi_z(f) = \lim_{T \rightarrow \infty} \frac{1}{2T} \mathbb{E} |Z_T(f)|^2 \quad ; \text{ Positive real function of } f$$

In Summary ;

$$n(t) = \operatorname{Re} \left\{ z(t) e^{j2\pi f_c t} \right\}$$

rep ①

$$n(t) = \underbrace{x(t)}_I \cos 2\pi f_c t - \underbrace{y(t)}_Q \sin 2\pi f_c t$$

$x(t)$  &  $y(t)$  are  
the quadrature component  
of  $n(t)$ .  
 $a(t)$  &  $\theta(t)$  are also  
stochastic processes.

rep ②

$$n(t) = a(t) \cos(2\pi f_c t + \theta(t))$$

rep ③

$$a(t) = \sqrt{x^2(t) + y^2(t)}, \quad \theta(t) = \tan^{-1} \frac{y(t)}{x(t)}$$

$$\Phi_n(f) = \frac{1}{2} [\Phi_z(f-f_c) + \Phi_z(-f-f_c)]$$

PSD rep.

Properties of  $x(t)$  &  $y(t)$ , the quadrature Components  
of bandpass stock Process  $n(t)$

Till now, we know 3 properties of  $x(t)$  &  $y(t)$

1)  $x(t)$  and  $y(t)$  are zero-mean jointly wss stochastic  
Processes.  $x(t)$  and  $y(t)$  are also low-pass stock. Proces  
 $z(t) = x(t) + jy(t)$

$$2) \quad \varphi_x(z) = \varphi_y(z) \implies \varphi_x(f) = \varphi_y(f)$$

$$3) \quad \varphi_{yx}(z) = -\varphi_{xy}(z) \quad \text{Jointly}$$

We know that the corr. function of any wss stochastic processes has the Hermitian property as below

$$\phi_{xy}(z) = \phi_{yx}(-z)$$

In general, Hermitian property

is

jointly real

$$\phi_{xy}^*(t+z, t) = \phi_{yx}(t, t+z)$$

wss stat. proc.

$$\phi_{xy}(z) = \phi_{yx}(z)$$

$$\left\{ \begin{array}{l} \phi_{xy}(z) = -\phi_{yx}(z) \\ \phi_{xy}(z) = \phi_{yx}(-z) \end{array} \right.$$

Property # 3

The property of any jointly wss  
real stat. processes.

$$\Rightarrow \phi_{yx}(z) = -\phi_{yx}(-z)$$

Property # 4 of  $x(t)$  &  $y(t)$

4) The joint corr. functions of  $x(t)$  &  $y(t)$  have  
Odd Symmetry, means

$$\left\{ \begin{array}{l} \varphi_{yx}(z) = -\varphi_{yx}(-z) \\ \varphi_{xy}(z) = -\varphi_{xy}(-z) \end{array} \right.$$

As you know, any odd symmetric functions, are equal to zero at the origin, means that for  $x(t) \neq y(t)$  we have

$$\left\{ \begin{array}{l} \left. \varphi_{xy}(z) \right|_{z=0} = \varphi_{xy}(0) = 0 \\ \left. \varphi_{yx}(z) \right|_{z=0} = \varphi_{yx}(0) = 0 \end{array} \right.$$

$$5) \quad \varphi_{xy}(z) \Big|_{z=0} = 0, \quad \varphi_{yx}(z) \Big|_{z=0} = 0$$

That means  $x(t)$  and  $y(t)$  are orthogonal at origin

$\tau = 0$ . As we know,  $x(t)$  and  $y(t)$  are zero-mean.

so at the origin ( $z=0$ )  $x(t)$  and  $y(t)$  are also uncorrelated.

\* If  $n(t)$  is a Gaussian stoch. Process, so  $z(t)$  is  
also a Complex Gaussian stoch. Process which show us  
that  $x(t)$  and  $y(t)$  are jointly Gaussian (normal) stoch.  
Processes. So, if  $n(t)$  is a Gaussian stoch. Process,  $x(t)$   
and  $y(t)$  are Independent at origin ( $\sigma = 0$ )

Because, in jointly Gaussian stoch. Processes, if two samples are uncorrelated we can conclude that these samples are also independent.

\* If for some stoch. processes  $n(t)$ , we have

$$\forall z, \phi_{yx}(z) = 0$$

$$\implies \varphi_z(z) = \varphi_x(z) + \underbrace{\varphi_{yx}(z)}_0 = \varphi_x(z) = \varphi_y(z)$$

$$\forall z, \varphi_{yx}(z) = 0 \quad \begin{matrix} \text{Property} \\ \neq z \end{matrix}$$

It means that  $\varphi_z(z)$  (Correlation function of  $z(t)$ ) is a real function of  $z$ .

So, the Fourier Transform of  $\varphi_z(z)$ ,  $FT_z \{ \varphi_z(z) \}$ ,  
is an even symmetric function of  $f$ .

$$\varphi_z(f) = FT_z \{ \varphi_z(z) \} \implies \varphi_z(-f) = \varphi_z(f)$$

$\varphi_z(z)$  is a real function of  $z$

So  $\varphi_z(f)$  is an even symmetric function  
of  $f$

In this case the relation between the PSD of bandpass stochastic process  $n(t)$  and the PSD of its even lowpass stochastic process will be as below

$$\varphi_n(f) = \frac{1}{2} [\varphi_z(f-f_c) + \varphi_z(-f-f_c)] = \frac{1}{2} [\varphi_z(f-f_c) + \varphi_z(f+f_c)]$$

$\varphi_z(f)$  have even symmetry

$\forall z, \varphi_{yy}(z) = 0 \Rightarrow \varphi_z(z) : \text{real}$

It means that if the Crr. function of a lowpass <sup>WSS</sup> stoch. process  
is real function of  $\omega$ , then its PSD has even symmetry Property.  
The reverse, is also True.

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for example in ASK modulated signals that their eqn. low-pass signals  
are real signal, we can see the even symmetry in their PSD functions.

NSS

Example : Band-Pass  $\checkmark$  white noise and its equ. low-pass process

As you know, a white noise stoch. process, has these characteristics.

$w(t)$  : white noise stoch process

$$1) m_w(t) = E[w(t)] = 0 \quad \forall t$$

$$2) R_w(\tilde{t} + \tau, \tilde{t}) = C_w(t + \tau, t) = g(\tilde{t}) S(\tau) = \begin{cases} 0 & \tau \neq 0 \\ g(t) & \tau = 0 \end{cases}$$

$(t_1 \neq t_2)$   
 $(t_1 = t_2)$

if  $g(t) = \text{cte} = g$  then  $w(t)$  is a wss white noise.

As you know, the PSD of any white noise stoch. processes is a constant function of  $f$ .

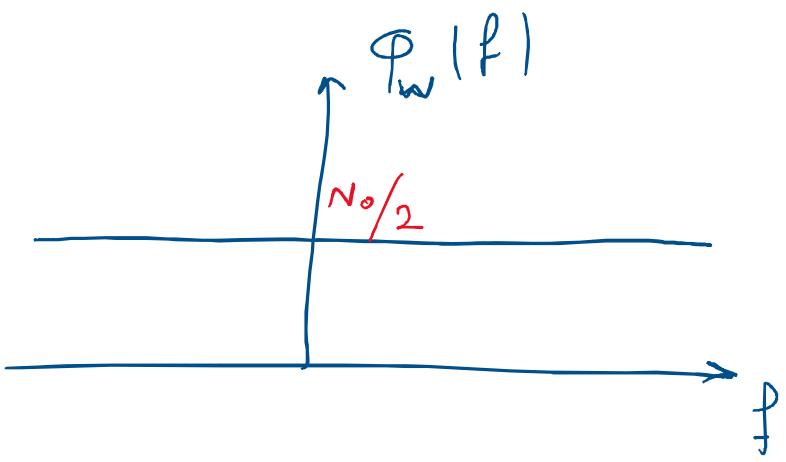
$$\varphi_w(f) = F\tau_\tau \left\{ \langle R_w(t+\tau, t) \rangle_t \right\} = F\tau_\tau \left\{ \langle g(t) \delta(\tau) \rangle_t \right\}$$

$$\varphi_w(f) = F\tau_\tau \left\{ \underbrace{\delta(\tau)}_{\text{constant } (N_0/2)} \langle g(t) \rangle_t \right\} = F\tau_\tau \left\{ \frac{N_0}{2} \delta(\tau) \right\} = \frac{N_0}{2} = \varphi_w(f)$$

As you see, white noise stochastic process, is not a

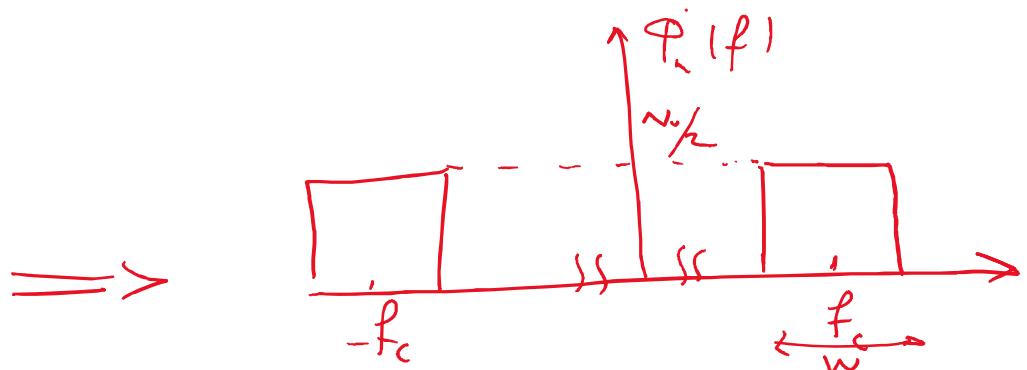
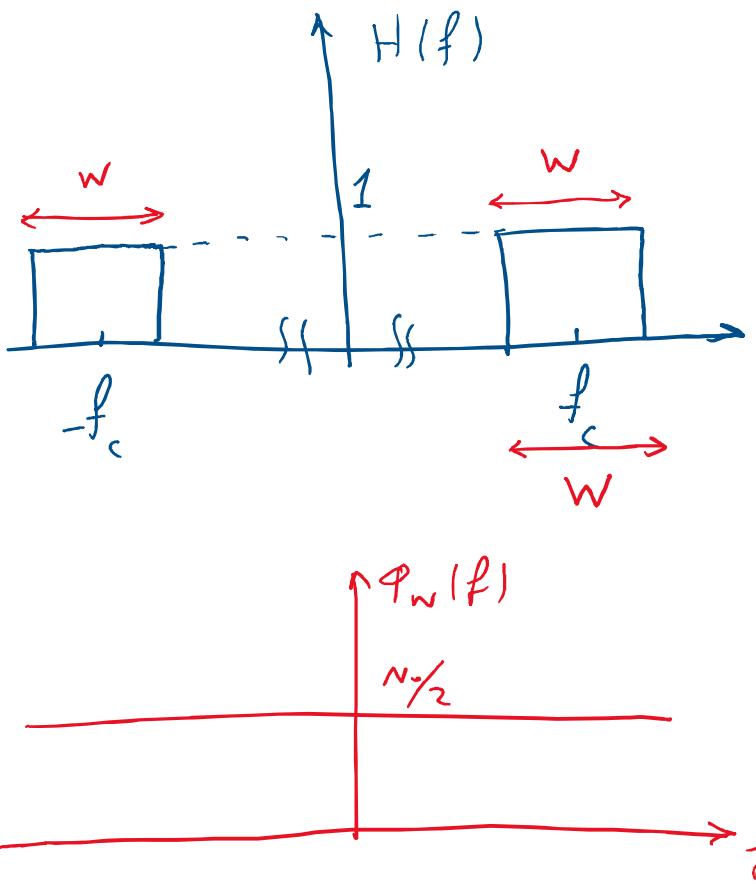
bandpass stoch. Process.

As you know, in Comm. Systems a specific frequency range and bandwidth is assigned for Comm. So, in the receiver, noise and

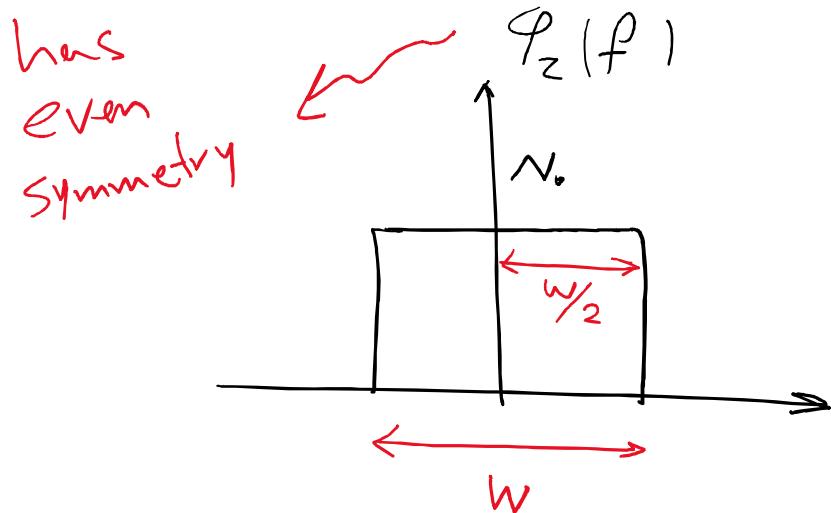
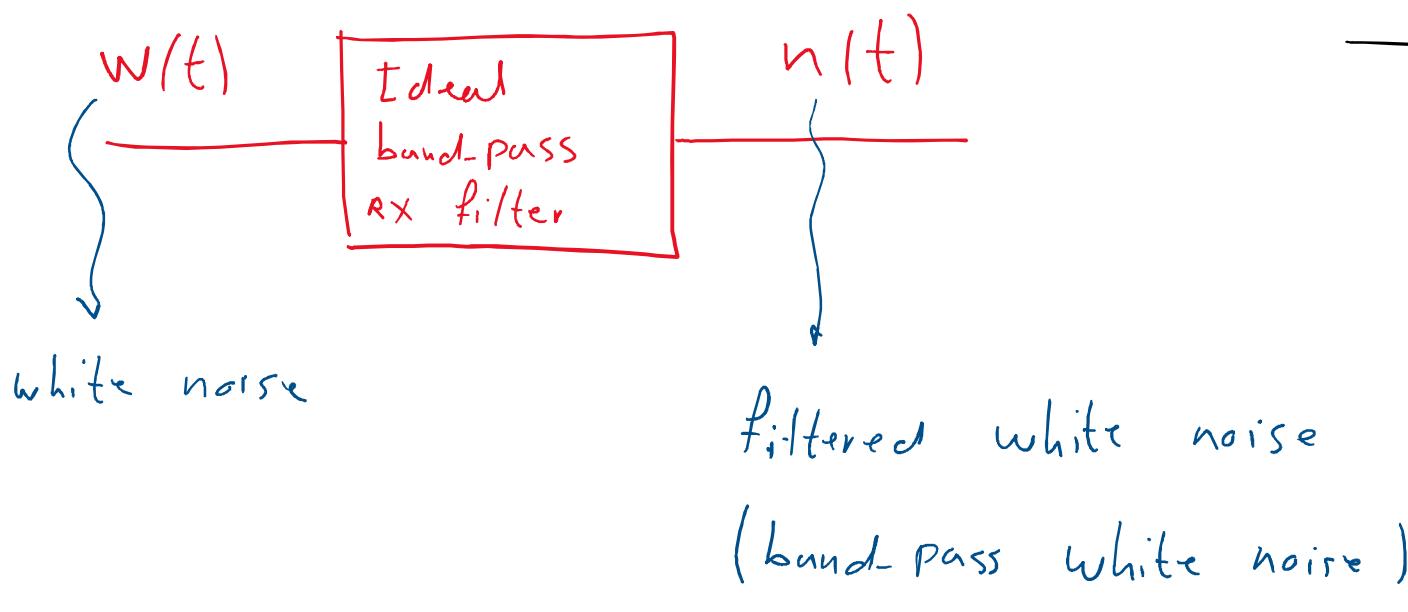


received signal, pass through an Rx filter, on the specific frequency band. For example an ideal Rx filter, may be as follows

$H(f)$  is the transfer function of an ideal band-pass Rx filter. White Noise is passed through this filter and in the output, we have a filtered white noise, call it, band-pass white noise.



$$\Phi_z(f) = \frac{1}{2} [\Phi_z(f-f_c) + \Phi_z(-f-f_c)]$$



$$\Phi_z(f) = \begin{cases} N_0 & |f| \leq \frac{w}{2} \\ 0 & \text{others.} \end{cases}$$

$$\Phi_z(f) = N_0 \left( \frac{f}{w} \right)$$

$$\begin{aligned} & \text{FT}^{-1} \\ \implies & \varphi_z(z) = w \operatorname{sinc} w z \quad (*) \end{aligned}$$

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Because,  $\varphi_z(f)$  has even symmetry we can conclude that  $\varphi_z(z)$  is real function of  $z$ . You can see this fact in the above relation (\*) and also we can write

$$\varphi_n[f] = \frac{1}{2} [\varphi_z(f-f_c) + \varphi_z(f+f_c)]$$